Optical activity in an isotropic gas of electrons with a preferred helicity

D. B. Melrose and J. I. Weise
School of Physics, University of Sydney, New South Wales 2006, Australia
(Received 10 March 2003; published 22 October 2003)

An isotropic gas of electrons with a preferred spin helicity is shown to be optically active. Simultaneous eigenfunctions of the Dirac Hamiltonian and the helicity operator are constructed and used to derive explicit expressions for vertex functions for helicity states. The (covariant) response tensor is calculated for an electron gas described in terms of a spin-dependent occupation number. The possibility of detecting optical activity in an electron gas is discussed briefly.

DOI: 10.1103/PhysRevE.68.046404 PACS number(s): 51.70.+f, 52.27.Ny, 12.20.−m

I. INTRODUCTION

The response (to an electromagnetic disturbance) of an isotropic classical electron gas can be decomposed into longitudinal and transverse parts, and these determine the dispersive properties of longitudinal and transverse wave modes, respectively. The most general response for an isotropic medium includes a third "rotatory" component [1], which is nonzero for optically active media, such as a solution of dextrose. The rotatory component removes the degeneracy in the two transverse modes, such that two oppositely circularly polarized modes have different refractive indices. Optical activity requires that the system have a preferred handedness. It has been pointed out that a rotatory component exists in a theory based on a Lagrangian with P- and CP-odd terms [2], where the asymmetry in the preferred handedness is intrinsic. Here we point out that a rotatory component exists in a simpler system: Electrons produced by decay of neutrons have a preferred spin helicity, and hence a preferred handedness. In this paper we calculate the response tensor for a helicity-dependent isotropic electron gas and show that it exhibits optical activity.

The derivation of the response tensor given here is intrinsically relativistic, and this is important even for what might otherwise be regarded as nonrelativistic electrons. The reason is that a physically relevant spin operator must commute with the Hamiltonian, and a nonrelativistic treatment using the Schrödinger-Pauli equation achieves this artificially by neglecting spin-orbit coupling. For electrons in motion it is necessary to choose a spin operator that commutes with the Dirac Hamiltonian, and a helicity operator exists that satisfies this criterion [3,4]. The electron states used in the derivation here are simultaneous eigenfunctions of the Dirac Hamiltonian and this helicity operator.

A conventional QED derivation of the response tensor for a relativistic quantum electron gas is analogous to the calculation for the vacuum polarization, with the electron propagator *in vacuo* replaced by the statistically averaged propagator in the electron gas [5]. This average involves the occupation numbers $n^{\epsilon}(\mathbf{p})$, for electrons $(\epsilon=+1)$ and positrons $(\epsilon=-1)$. This formalism involves taking a trace over a product of Dirac matrices, and it applies only to unpolarized electrons, in the sense that $n^{\epsilon}(\mathbf{p})$ is to be interpreted as $\overline{n^{\epsilon}}(\mathbf{p})=\frac{1}{2}[n_{+}^{\epsilon}(\mathbf{p})+n_{-}^{\epsilon}(\mathbf{p})]$, where $n_{s}^{\epsilon}(\mathbf{p})$ is the occupation number for the spin state $s=\pm 1$. The spin dependence is included here by writing the general expression for the response tensor in terms of spin-dependent vertex functions, analogous to those used to treat QED processes in a magnetized plasma [6,7].

In Sec. II a covariant form for the linear response tensor is decomposed into longitudinal, transverse, and rotatory parts. The response tensor for an arbitrary, spin-dependent electron gas is then written down and is separated into non-spin-dependent and spin-dependent parts. In Sec. III the helicity eigenstates are written down, the vertex function is evaluated for the helicity states, and the spin-dependent part of the response tensor is evaluated explicitly and shown to be of the rotatory form identified in Sec. II. In Sec. IV the possibility of detecting optical activity in an electron gas is discussed. Natural units, with $\hbar=c=1$ are used, except where stated otherwise.

II. RESPONSE TENSOR FOR AN UNMAGNETIZED ELECTRON GAS

A covariant form for the linear response tensor $\Pi^{\mu\nu}(k)$ relates the induced 4-current $J^{\mu}=\Pi^{\mu\nu}(k)A_{\nu}(k)$ to the 4-potential of the disturbance, $A(k)$, where k is the wave 4-vector. Charge continuity and gauge invariance require

$$k_{\mu}\Pi^{\mu\nu}(k)=0, \quad k_{\nu}\Pi^{\mu\nu}(k)=0, \tag{1}$$

respectively.

A. Separation into longitudinal, transverse, and rotatory parts

For an isotropic medium, with U being the 4-velocity of the rest frame of the medium, the most general form of the response tensor e.g., Ref. [2] is

$$\Pi^{\mu\nu}(k)=\Pi^{L}(k)L^{\mu\nu}(k,U)+\Pi^{T}(k)T^{\mu\nu}(k,U) +\Pi^{R}(k)R^{\mu\nu}(k,U), \tag{2}$$

where $\Pi^{L}(k)$, $\Pi^{T}(k)$, and $\Pi^{R}(k)$ are invariants that describe the longitudinal, transverse, and rotatory responses, respectively. The longitudinal tensor can be expressed as the outer product of a longitudinal 4-vector, $L(k,U)$, with itself. One requires $kL(k,U)=0$ to ensure that Eq. (1) is satisfied. A specific choice is

$$L^{\mu\nu}(k,U)=-L^{\mu}(k,U)L^{\nu}(k,U),$$

$$L^{\mu}(k,U)=\frac{kU\,k^{\mu}-k^2U^{\mu}}{kU\,[(kU)^2-k^2]^{1/2}},\tag{3}$$

which corresponds to a normalization such that in the rest frame, $U=[1,\mathbf{0}]$, one has $L(k,U)=[|\mathbf{k}|/\omega,\mathbf{k}/|\mathbf{k}|]$. The transverse tensor

$$T^{\mu\nu}(k,U)=\frac{(kU)^2}{k^2}L^{\mu}(k,U)L^{\nu}(k,U)+g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{k^2},\tag{4}$$

where $g^{\mu\nu}$ is the metric tensor (signature -1), is normalized such that in the rest frame it corresponds to the unit transverse second rank tensor. The rotatory tensor is defined by

$$R^{\mu\nu}(k,u)=i\epsilon^{\mu\nu\rho\sigma}L_{\rho}(k)U_{\sigma},\tag{5}$$

where the completely asymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ has $\epsilon^{0123}=1$. One has

$$\Pi^{L}(k)=\frac{(kU)^4}{k^4}L_{\sigma\nu}(k,U)\Pi^{\sigma\nu}(k),$$

$$\Pi^{T}(k)=\frac{1}{2}T_{\sigma\nu}(k,U)\Pi^{\sigma\nu}(k),$$

$$\Pi^{R}(k)=-\frac{1}{2}R_{\sigma\nu}(k,U)\Pi^{\sigma\nu}(k).\tag{6}$$

B. Response tensor for an arbitrary electron gas

A derivation using QED [8] leads to the following general expression for the response tensor:

$$\Pi^{\mu\nu}(k)=-e^2\sum_{\epsilon,s,\epsilon',s'}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}(2\pi)^3 \times\delta^3(\epsilon'\mathbf{p}'-\epsilon\mathbf{p}+\mathbf{k})\frac{\epsilon n_s^{\epsilon}(\mathbf{p})-\epsilon' n_{s'}^{\epsilon'}(\mathbf{p}')}{\omega-\epsilon\varepsilon+\epsilon'\varepsilon'+i0} \times[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu}[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{*\nu},\tag{7}$$

with $\varepsilon=(m^2+\mathbf{p}^2)^{1/2}$, $\varepsilon'=(m^2+\mathbf{p}'^2)^{1/2}$. The electron gas is described as a statistical distribution of electrons ($\epsilon=+$) and positrons ($\epsilon=-$) having occupation numbers $n_s^{\epsilon}(\mathbf{p})$, with $s=\pm1$ being the spin eigenvalue. The vertex functions in Eq. (7) are defined as

$$[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu}=\frac{\overline{u}_{s'}^{\epsilon'}(\epsilon'\mathbf{p}')\gamma^{\mu}u_s^{\epsilon}(\epsilon\mathbf{p})}{\sqrt{2\varepsilon'}\sqrt{2\varepsilon}},\tag{8}$$

where γ^{μ} are the Dirac matrices and $u_s^{\epsilon}(\epsilon\mathbf{p})$ are the electron eigenfunctions.

C. Separation into spin-independent and spin-dependent parts

A separation into spin-independent and spin-dependent parts involves separating the occupation number into spin-averaged and spin-specific terms, by writing

$$n^{\epsilon}(\mathbf{p})=\frac{1}{2}[n_+^{\epsilon}(\mathbf{p})+n_-^{\epsilon}(\mathbf{p})],$$

$$\Delta n^{\epsilon}(\mathbf{p})=\frac{1}{2}[n_+^{\epsilon}(\mathbf{p})-n_-^{\epsilon}(\mathbf{p})],\tag{9}$$

so that one has

$$\sum_{s's} n_s^{\epsilon}(\mathbf{p})[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu}[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{*\nu} =n^{\epsilon}(\mathbf{p})[\overline{\alpha}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]+\Delta n^{\epsilon}(\mathbf{p})[\Delta\alpha^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})],$$

$$\sum_{s's} n_{s'}^{\epsilon'}(\mathbf{p}')[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu}[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{*\nu} =n^{\epsilon'}(\mathbf{p}')[\overline{\alpha}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]+\Delta n^{\epsilon'}(\mathbf{p}') \times[\Delta'\alpha^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})],\tag{10}$$

in Eq. (7), with

$$[\overline{\alpha}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu\nu}=\sum_{s,s'}[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu}[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{*\nu},$$

$$[\Delta\alpha_{q'q}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu\nu}=\sum_{s,s'} s[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu} \times[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{*\nu},$$

$$[\Delta'\alpha_{q'q}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu\nu}=\sum_{s,s'} s'[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu} \times[\Gamma_{s's}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{*\nu}.\tag{11}$$

The linear response tensor in Eq. (7) then separates into a spin-independent part $\Pi_{\rm in}^{\mu\nu}$ and a spin-dependent part $\Pi_{\rm sd}^{\mu\nu}$, with $\Pi^{\mu\nu}(k)=\Pi_{\rm in}^{\mu\nu}(k)+\Pi_{\rm sd}^{\mu\nu}(k)$. For the spin-independent part, the sum over spins may be carried out by standard techniques without introducing any spin operator. One has

$$\sum_{s,s'}[\overline{\alpha}^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu\nu}=\frac{F^{\mu\nu}(\epsilon\tilde{p},\epsilon'\tilde{p}')}{\epsilon\epsilon'\varepsilon\varepsilon'},\tag{12}$$

$$F^{\mu\nu}(P,P')=\frac{1}{4}\mathrm{Tr}[\gamma^{\mu}(\not P+m)\gamma^{\nu}(\not P'+m)] =P^{\mu}P'^{\nu}+P'^{\mu}P^{\nu}+g^{\mu\nu}(m^2-PP'),\tag{13}$$

$\tilde{p}=[\varepsilon,\mathbf{p}]$, $\tilde{p}'=[\varepsilon',\mathbf{p}']$. The resulting spin-independent part of the response tensor can be rewritten in a variety of known forms for an unpolarized electron gas. A form that follows directly from Eq. (12) is

$$\Pi^{\mu\nu}_{\rm in}(k)=2e^2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{\epsilon}\frac{n^{\epsilon}(\mathbf{p})}{\varepsilon}\left[\frac{F^{\mu\nu}(\epsilon p,\epsilon p-k)}{k^2-2\epsilon pk}+\frac{F^{\mu\nu}(\epsilon p+k,\epsilon p)}{k^2+2\epsilon pk}\right]. \tag{14}$$

Apart from minor differences in notation, form (14) is that written down by Ref. [9]. The spin-dependent part of the response tensor is

$$\Pi^{\mu\nu}_{\rm sd}(k)=-e^2\sum_{\epsilon,\epsilon'}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\epsilon\Delta n^{\epsilon}(\mathbf{p})}{\omega-\epsilon\varepsilon+\epsilon'\varepsilon'+i0}\times[\Delta\alpha^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu\nu}+e^2\sum_{\epsilon,\epsilon'}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\times\frac{\epsilon'\Delta n^{\epsilon'}(\mathbf{p}')}{\omega-\epsilon\varepsilon+\epsilon'\varepsilon'+i0}[\Delta'\alpha^{\epsilon'\epsilon}(\mathbf{p}',\mathbf{p},\mathbf{k})]^{\mu\nu}. \tag{15}$$

The functions $[\Delta\alpha^{\epsilon'\epsilon}]^{\mu\nu}$, and $[\Delta'\alpha^{\epsilon'\epsilon}]^{\mu\nu}$ depend explicitly on the choice of spin operator.

III. RESPONSE OF A HELICITY-DEPENDENT ELECTRON GAS

The tensors $[\overline{\alpha}^{\epsilon'\epsilon}]^{\mu\nu}$, $[\Delta\alpha^{\epsilon'\epsilon}]^{\mu\nu}$, and $[\Delta'\alpha^{\epsilon'\epsilon}]^{\mu\nu}$, introduced in Eq. (11), are evaluated here in the case where the spin operator is identified as the helicity.

A. Helicity eigenstates

An acceptable spin operator must commute with the Dirac Hamiltonian [3,4,10], and the helicity operator $\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}$, satisfies this criterion. The coordinate representation implies $\hat{\mathbf{p}}=-i\partial/\partial\mathbf{x}$, and $\boldsymbol{\sigma}$ denotes the Pauli matrices in the 4×4 Dirac spin space. Assuming a plane wave solution $\propto\exp[-i\epsilon(\varepsilon t-\mathbf{p}\cdot\mathbf{x})]$, and the 3-momentum in cylindrical polar coordinates, $\mathbf{p}=(p_\perp\cos\phi,p_\perp\sin\phi,p_z)$, the eigenvalues of the helicity operator are sp with $s=\pm$ and $p=|\mathbf{p}|=(p_z^2+p_\perp^2)^{1/2}$. A specific choice of eigenstates, for a convenient choice of phase, is

$$u^{\epsilon}_s(\epsilon\mathbf{p})=\frac{1}{\sqrt{2p}}\begin{pmatrix}\sqrt{\varepsilon+\epsilon m}\,\sqrt{p+\epsilon sp_z}\,e^{-i\phi/2}\\ s\epsilon\sqrt{\varepsilon+\epsilon m}\,\sqrt{p-\epsilon sp_z}\,e^{i\phi/2}\\ s\epsilon\sqrt{\varepsilon-\epsilon m}\,\sqrt{p+\epsilon sp_z}\,e^{-i\phi/2}\\ \sqrt{\varepsilon-\epsilon m}\,\sqrt{p-\epsilon sp_z}\,e^{i\phi/2}\end{pmatrix}. \tag{16}$$

The vertex function (8) for the helicity eigenstates becomes

$$[\Gamma^{\epsilon'\epsilon}_{s's}(\mathbf{p}',\mathbf{p})]^{\mu}=\frac{1}{4(p'\varepsilon'p\varepsilon)^{1/2}}\begin{pmatrix}[\alpha'_+\alpha_++\Sigma\alpha'_-\alpha_-][\beta'_+\beta_+e^{-i(\phi-\phi')/2}+\Sigma\beta'_-\beta_-e^{i(\phi-\phi')/2}]\\ [\alpha'_+\alpha_-+\Sigma\alpha'_-\alpha_+][\beta'_+\beta_-e^{i(\phi+\phi')/2}+\Sigma\beta'_-\beta_+e^{-i(\phi+\phi')/2}]\\ -i[\alpha'_+\alpha_-+\Sigma\alpha'_-\alpha_+][\beta'_+\beta_-e^{i(\phi+\phi')/2}-\Sigma\beta'_-\beta_+e^{-i(\phi+\phi')/2}]\\ s\epsilon[\alpha'_+\alpha_-+\Sigma\alpha'_-\alpha_+][\beta'_+\beta_+e^{-i(\phi-\phi')/2}-\Sigma\beta'_-\beta_-e^{i(\phi-\phi')/2}]\end{pmatrix}, \tag{17}$$

$$\alpha_\pm=(\varepsilon\pm\epsilon m)^{1/2},\quad \beta_\pm=(p\pm\epsilon sp_z)^{1/2},\quad \Sigma=\epsilon's'\epsilon s, \tag{18}$$

and similarly $\alpha'_\pm=(\varepsilon'\pm\epsilon'm)^{1/2}$ and $\beta'_\pm=(p'\pm\epsilon's'p'_z)^{1/2}$.

B. The response tensor for the helicity eigenstates

The nonzero components of $[\Delta\alpha^{\epsilon'\epsilon}]^{\mu\nu}$ and $[\Delta'\alpha^{\epsilon'\epsilon}]^{\mu\nu}$, cf. Eq. (11), give the helicity-dependent part of the response tensor:

$$\Pi^{\mu\nu}_{\rm sd}(k)=-ie^2k^2\sum_{\epsilon}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\Delta n^{\epsilon}(\mathbf{p})}{(pk)^2-(k^2/2)^2}b^{\mu\nu}(k,p),$$

$$b^{01}=\frac{k_zp_y-k_yp_z}{|\mathbf{p}|},\quad b^{02}=\frac{k_xp_z-k_zp_x}{|\mathbf{p}|},\quad b^{03}=\frac{k_yp_x-k_xp_y}{|\mathbf{p}|},$$

$$b^{12}=\frac{\omega\varepsilon p_z-k_z|\mathbf{p}|^2}{|\mathbf{p}|\varepsilon},\quad b^{13}=\frac{k_y|\mathbf{p}|^2-\omega\varepsilon p_y}{|\mathbf{p}|\varepsilon},$$

$$b^{23}=\frac{\omega\varepsilon p_x-k_x|\mathbf{p}|^2}{|\mathbf{p}|\varepsilon}, \tag{19}$$

with $pk=\omega\varepsilon-\mathbf{p}\cdot\mathbf{k}$, $k^2=\omega^2-|\mathbf{k}|^2$.

C. Isotropic distribution

It follows directly from Eq. (19) that the spin-dependent contribution of an isotropic helicity-dependent electron gas is rotatory. To see this, choose $\mathbf{k}$ along the z axis ($k_x=0$, $k_y=0$, $k_z=|\mathbf{k}|$) in the rest frame of the electron gas. Then in the denominator of the integrand in Eq. (19) one has $\mathbf{p}\cdot\mathbf{k}=p_z|\mathbf{k}|$, so that the integrals over terms proportional to p_x or p_y in $b^{\mu\nu}$ give zero. The only nonzero term in Eq. (19) is then $b^{12}=-b^{21}$. By inspection, the only nonzero component of $R^{\mu\nu}$ for $\mathbf{k}$ along the z axis in the rest frame is the 12-component, completing the proof.

The invariants Π^L, Π^T, and Π^R can each be reduced to a single integral by performing the angular integrals in Eq. (19). Various forms for Π^L and Π^T are known, and Π^R can be written in a corresponding variety of forms. One form is

$$\Pi^R(k)=-\frac{e^2}{(2\pi)^2}\int_0^\infty dp\,\frac{\Delta\overline{n}(p)}{\varepsilon|\mathbf{k}|^2}\Bigg\{[\omega\varepsilon(\omega\varepsilon-k^2/2)-p^2|\mathbf{k}|^2]\times\ln\left[\frac{\omega\varepsilon-k^2/2+p|\mathbf{k}|}{\omega\varepsilon-k^2/2-p|\mathbf{k}|}\right]-[\omega\varepsilon(\omega\varepsilon+k^2/2)-p^2|\mathbf{k}|^2]\times\ln\left[\frac{\omega\varepsilon+k^2/2+p|\mathbf{k}|}{\omega\varepsilon+k^2/2-p|\mathbf{k}|}\right]\Bigg\}, \qquad (20)$$

with $\Delta\overline{n}(p)=\Delta n^+ + \Delta n^-$.

D. Thermal (Jüttner) distribution

A nondegenerate relativistic thermal electron gas is described by the Jüttner distribution

$$\overline{n}(p)=\frac{\pi^2 n\rho e^{-\rho\gamma}}{K_2(\rho)m^3}, \qquad (21)$$

where n is the number density, $\rho=m/T$ is the inverse temperature in units of the rest energy of the particle, $\gamma=(1-v^2)^{-1/2}$ is the Lorentz factor and $K_2(\rho)$ is a Macdonald (modified Bessel) function. In the nonquantum limit, Π^L and Π^T for a Jüttner distribution may be written in terms of the relativistic plasma dispersion function [11]

$$T(z,\rho)=\int_{-1}^1 dv\,\frac{e^{-\rho\gamma}}{v-z} \qquad (22)$$

and its derivative with respect to $z=\omega/|\mathbf{k}|$. The general quantum expressions for the longitudinal and transverse parts may also be written in terms of $T(z,\rho)$ [12]. However, the rotatory part is an odd function of $v=p/\gamma m$ and hence it cannot be expressed in terms of $T(z,\rho)$ which relies on the response being an even function. By introducing a class of functions

$$V_n(z,\rho)=\int_0^1 dv\ \gamma^n\, e^{-\rho\gamma}\ln\left(\frac{z-v}{z}\right), \qquad (23)$$

$T(z,\rho)$ and functions derived from it may be expressed in terms of even combinations $V_n(z,\rho)+V_n(-z,\rho)$, whereas here odd combinations appear. Specifically, $\Pi^R(k)$ may be written in the form

$$\Pi^R(k)=\frac{\omega_p^2}{\mu_0}\Delta\frac{\rho}{4K_2(\rho)}\frac{\omega}{m}(1-z^2)\times\left\{\frac{2}{z\rho}e^{-\rho}+V_3(z,\rho)-V_3(-z,\rho)\right\}, \qquad (24)$$

where $\Delta=\Delta\overline{n}(\mathbf{p})/\overline{n}(\mathbf{p})$ is the fractional excess of electrons with positive helicity.

E. Nonrelativistic limit

The difference in the refractive indices between the left and right hand polarized modes is determined by Π^R/Π^T. Consider a nonrelativistic thermal distribution of electrons at a temperature $T=mV_e^2$. Assuming phase speeds much greater than the thermal speed, $z\gg V_e$, one has, in ordinary units,

$$\Pi^T(k)=-\frac{\omega_p^2}{\mu_0},\quad \Pi^R(k)=\Delta\frac{\omega_p^2}{\mu_0}\frac{\omega^2-|\mathbf{k}|^2c^2}{|\mathbf{k}|^2c^2}\frac{3\hbar|\mathbf{k}|V_e}{(2\pi)^{1/2}mc^2}. \qquad (25)$$

The refractive indices for the two modes, labeled $\pm$, are

$$n_\pm^2=1-\frac{\omega_p^2}{\omega^2}\left[1\pm\Delta\frac{\omega_p^2}{\omega^2}\frac{3\hbar\omega}{(2\pi)^{1/2}mc^2}\frac{V_e}{c}\right], \qquad (26)$$

where the term proportional to Δ is assumed small in making the approximation $\omega^2-|\mathbf{k}|^2c^2=\omega_p^2$ to lowest order in Δ.

IV. DISCUSSION AND CONCLUSIONS

The optical activity associated with a gas of electrons with a specific helicity is an intrinsically quantum mechanical effect, and hence is intrinsically small. In principle, the effect is observable in terms of the rotation of the plane of polarization as radiation propagates through the electron gas. Significant rotation requires propagation over a length L that satisfies $|n_+-n_-|L\omega/c\gtrsim 1$. Using Eq. (26), this condition requires $\hbar\omega_p^4V_eL/\omega^2mc^4\gtrsim 1$. In order of magnitude, this requires $(\omega_p^2/\omega^2)(V_e/c)(nL/10^{27}\ \mathrm{m}^{-2})\gtrsim 1$, where n is the number density per cubic meter and L is in meters. A dense plasma, a long path length, and a frequency not too much greater than the plasma frequency are needed for the effect to be readily observable, and these are very difficult conditions to satisfy. A severe constraint is that in a dense plasma, collisions depolarize the electrons on a collisional time scale, seemingly precluding observation of the effect in the laboratory.

Another complication is that in the presence of even a very weak magnetic field, the Faraday effect can swamp the effect of the optical activity. The difference in refractive indices due to the Faraday effect $|n_+-n_-|$ is of order $\omega_p^2\omega_B/\omega^3$, where $\omega_B=eB/m$ is the cyclotron frequency. The ratio of the cyclotron frequency to the plasma frequency needs to satisfy $\omega_B/\omega_p\lesssim(\hbar\omega_p/mc^2)(V_e/c)$ for the Faraday effect to be smaller than the effect of optical activity, and this is an extremely difficult condition to satisfy. In principle, one can separate the Faraday effect and optical activity by reflecting the radiation after it has passed through the electron gas once, so that it retraces its ray path back to its source. The rotations of the plane of polarization along the direct and reflected ray paths has opposite signs for the Faraday effect and the same sign for optical activity. Hence, the Faraday effect can be made to cancel while doubling the effect of optical activity. This suggests that the optical activity could be separated from the Faraday effect in an arrangement involving multiple reflections so that the ray traverses the same ray path many times. Nevertheless, the Faraday effect provides another severe restriction on the possibility of observing the optical activity.

We conclude that although an electron gas favoring electrons with a specific spin helicity exhibits optical activity in principle, it does not seem feasible to detect this effect using familiar techniques.

[1] D.B. Melrose and R.C. McPhedran, *Electromagnetic Processes in Dispersive Media* (Cambridge University Press, Cambridge, 1991).

[2] J.F. Nieves and P.B. Pal, Phys. Rev. D **39**, 652 (1989).

[3] A.A. Sokolov and I.M. Ternov, *Synchrotron Radiation* (Pergamon Press, Oxford, 1968).

[4] A.A. Sokolov, and I.M. Ternov, *Radiation from Relativistic Electrons* (American Institute of Physics, New York, 1986).

[5] V.N. Tsytovich, Zh. Eksp. Teor. Phys. **40**, 1775 1961 [Sov. Phys. JETP, **13**, 1249 (1961)].

[6] D.B. Melrose and A.J. Parle, Aust. J. Phys. **36**, 755 (1983).

[7] D.B. Melrose and K. Russell, J. Phys. A **35**, 135 (2002).

[8] D.B. Melrose and A.J. Parle, Aust. J. Phys. **36**, 799 (1983).

[9] L.M. Hayes and D.B. Melrose, Aust. J. Phys. **37**, 615 (1984).

[10] D.M. Fradkin and R.H. Good, Jr., Rev. Mod. Phys. **33**, 343 (1961).

[11] B.B. Godfrey, B.S. Newberger, and K.A. Taggart, IEEE Trans. Plasma Sci. **3**, 60 (1975).

[12] D.B. Melrose, Aust. J. Phys. **35**, 41 (1982).